A characterization of quasi-rational polygons

Nicolas Bedaride

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ABSTRACT

The aim of this paper is to study quasi-rational polygons related to the outer billiard. We compare different notions introduced in [GS92] and [Sch09] and make a synthesis of those.

1 Introduction

The outer billiard map is a transformation T of the exterior of a planar convex bounded domain D defined as follows: $T(M) = N$ if the segment MN is tangent to the boundary of D at its midpoint, and D lies at the right of MN. The outer billiard map is not defined if the tangent segment MN shares more than one point with the boundary of D. In the case where P is a convex polygon; the set of points for which T or any of its iterations is not defined is contained in a countable union of lines and has zero measure. The dual billiard map has been introduced by Neumann in [Neu59] as a toy model for the planet orbits. One of the most interesting questions was whether the orbits of T might escape to infinity for a polygonal domain D.

Two particular classes of polygons have been introduced by Kolodziej et al. in several articles, see [Koł89, GS92, VS87]. These classes are named rational and quasi-rational polygons and contain all the regular polygons. A rational polygon has vertices on a lattice of $\mathbb{R}^2$. They prove that every orbit outside a polygon in this class is bounded. Every regular polygon is a quasi-rational polygon, and it is not a rational polygon except if there are $3, 4$ or 6 edges. In the case of the regular pentagon, Tabachnikov completely described the dynamics of the outer billiard map in terms of symbolic dynamics, see [Tab95b]. He proves that some orbits are bounded and non periodic. The symbolic coding of this map has been given in [BC11] for a regular polygon with $3, 4, 5, 6$ and 10 edges.

For non quasi-rational polygons, there is no general study. The case of trapezoids has been studied. The set of trapezoids can be parametrized up

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to affinity by one parameter. For an irrational parameter, it is not a quasi-rational polygon, and the proof of [GS92] cannot be used for a polygon with parallel sides. Nevertheless, Li proved that all the orbits of the outer billiard map are bounded (this theorem is also proved by Genin) see [Li09] and [Gen08]. Recently Schwartz described a family of quadrilaterals, named kites, for which there exists unbounded orbits, see [Sch07] and [Sch09]. In these papers Schwartz introduces many tools in order to study the dynamics. These tools can also be used in the case of regular polygons, see [Sch10].

In this article we investigate the case of quasi-rational polygons. The main achievements of the paper consist of a synthesis of results of [GS92] and the notions introduced by Schwartz. These links allow us to give a new characterization of this class and to give some simple conditions which guarantee the quasi rationality.

Remark 1. *In this article, P is a polygon with n vertices without parallel edges, see last section for some comments. All the figures correspond to the same polygon.*

2 Overview of the paper

First we recall usual definitions about dual billiard in Section 3 and introduce our definition of quasi-rational polygon. Next, in Section 5, we show that our definition is equivalent to the old one of [GS92] and also similar to [Sch09]. In Section 6 we prove the classical theorem on quasi-rational polygon using our definition. Finally in Section 8 we use our definition to obtain new results on quasi-rational polygons.

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3 Outer billiard

We refer to [Tab95a] or [GS92]. We consider a convex polygon P in $\mathbb{R}^2$ with n vertices. Let $\overline{P} = \mathbb{R}^2 \setminus P$ be the complement of P.
We fix an orientation on $\mathbb{R}^2$. We will define the outer billiard map off a countable union of lines. The map will be defined for all time.

For a point $M \in \overline{P}$, there are two half-lines R, R' emanating from M and tangent to P, see Figure 1. Assume that the oriented angle R, R' has positive measure. Denote by A^+, A^- the tangent points on R respectively R'. We say that A^+ is the vertex **associated** to M.

Definition 1. *The outer billiard map is the map T defined as follows:*

$$T(M) = r_{A^+}(M)$$

where r_{A^+} is the reflection about A^+.

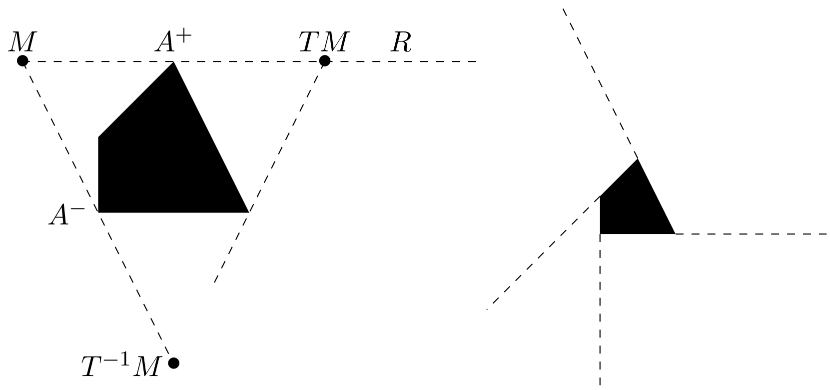

Figure 1: The outer billiard map

Definition 2. *A polygon P is said to be rational if the vertices of P are on a lattice of $\mathbb{R}^2$.*

We refer to [Sch09]. Consider a polygon without parallel edges. Assume the edges are oriented counterclockwise sense (while T is oriented clockwise). For each edge, consider the vertex of P furthest from the line supporting the edge. It is unique by convexity and assumption. Then denote by V the vector equal to twice the vector between the final vertex of the edge and this vertex. A **strip** is the band formed by the line supporting the edge and V, see Figure 2. It is denoted (Σ, V) or Σ if clear from the context. We index them with respect to the slopes of the sides of the polygon, this gives the sequences $(\Sigma_i, V_i)_{1 \leq i \leq n}$.

Definition 3. *Let $\alpha_1, \dots, \alpha_n$ be non zero real numbers, we say that $(\alpha_1, \dots, \alpha_n)$ are commensurate if $\frac{\alpha_2}{\alpha_1}, \dots, \frac{\alpha_n}{\alpha_{n-1}}$ are rational numbers.*

We denote by $u \wedge v$ the cross product of two vectors u, v of the plane. It is a vector of $\mathbb{R}^3$ orthogonal to the plane with only one non zero coordinate. The absolute value of this coordinate is denoted $|u \wedge v|$.

Definition 4. *The polygon P is quasi-rational if and only if $(|V_1 \wedge V_2|, \dots, |V_n \wedge V_1|)$ are commensurates.*

For example, consider the polygon with vertices A, B, C, D, see Figure 2. The vectors are equal to: $V_1 = 2\vec{CB}, V_2 = 2\vec{AC}, V_3 = 2\vec{BD}, V_4 = 2\vec{BA}$.

4 Unfolding

In this Section we recall the notion of unfolding introduced in [GS92]. This notion is used to transform the outer billiard map in a piecewise translation map defined on cones.

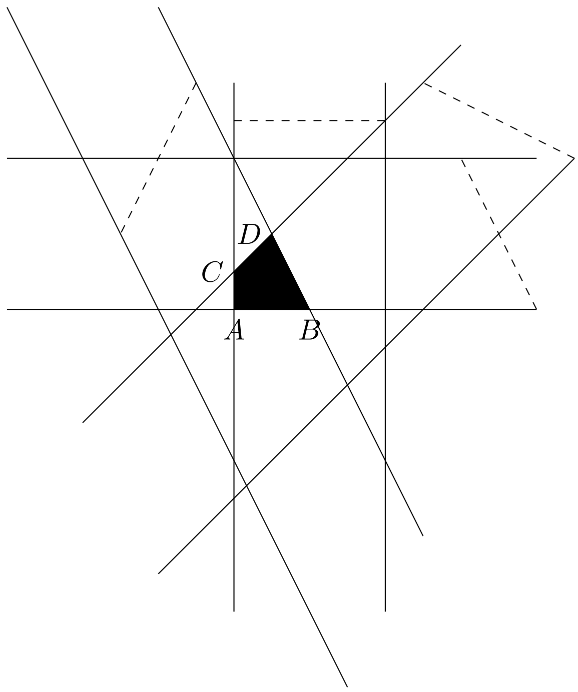

Figure 2: Polygon and strips

4.1 Definitions

Consider two vertices A, B of P, and the images of P by the rotations r_A, r_B of angle π. They are equal up to translation. Denote by $\tilde{P}$ one of these polygons. Let $\mathcal{S}_P$ be the set of those polygons in $\mathbb{R}^2$ that are images by a translation by P or $\tilde{P}$. Let M be a point in $\overline{P}$, define the following bijective map.

$$\begin{array}{rccc} \pi_M & : & \mathcal{S}_P & \to & \mathbb{R}^2 \times \{-1, 1\} \\ & & Q & \mapsto & (A, \varepsilon) \end{array}$$

Consider the image of M by the outer billiard map outside Q. It is obtained by a rotation of angle π centered at a vertex of Q. Let A be this vertex of Q. Moreover we take $\varepsilon = 1$ if Q is a translate of P, $\varepsilon = -1$ if Q is a translate of $\tilde{P}$. We say A is associated to M for Q. It is clear that π_M is a bijection.

We define a new map called **the unfolding** of the dual billiard map.

$$\begin{array}{rccc} \tilde{T} & : & \mathbb{R}^2 \times \{-1, 1\} & \to & \mathbb{R}^2 \times \{-1, 1\} \\ & & (A, \varepsilon) & \mapsto & (A', -\varepsilon) \end{array}$$

The ordered pair (A, ε) comes from a polygon Q via the map π_M. Consider the polygon Q' image of the polygon Q by a rotation of angle π of center A, see Figure 3. The point A' is the vertex associated to M for Q'.

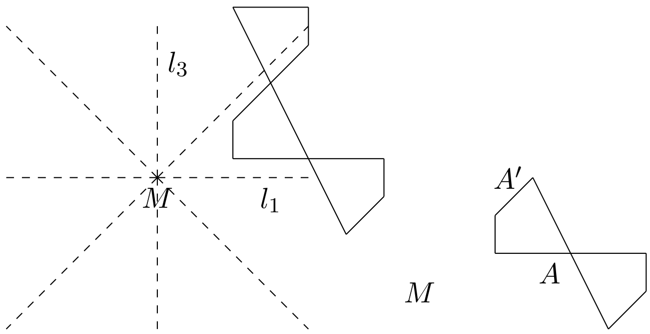

Figure 3: Necklace dynamics

The dynamics of this map is related to the outer billiard map by the following result. In what follows we will also denote by $\tilde{T}$ the projection of $\tilde{T}$ to $\mathbb{R}^2$.

Definition 5. *Denote by $(l_i)_{i \leq n}$ the lines passing through M and parallel to the edges of P. They defines $2n$ cones $(C_i)_{1 \leq i \leq 2n}$. The boundary of each cone is made of two half lines denoted R_i, R_{i+1}.*

Proposition 1. *[GS92] We have:*

- *The sequence $(T^k(M))_k$ is bounded (resp. periodic) if and only if there exists a point $Q \in \mathbb{R}^2 \times \{-1, 1\}$ such that the orbit of Q is bounded (resp. periodic) for $\tilde{T}$.*
- *For every cone C_i, there exists a vector a_i such that if $A, \tilde{T}A \in C_i$, then the restriction of $\tilde{T}$ to a cone is a translation of vector a_i. Moreover we have for every integer i, $a_{n+i} = -a_i$.*
- *There exists a polygon P^* with $2n$ edges, with vertices on $C_1, \ldots, C_n$ such that each side is parallel to some a_i.*

The sides of P^* will be denoted $v_i^*, i = 1 \ldots 2n$.

4.2 Some results

Here we explain how to find the vectors $a_1, \ldots, a_n$.

Definition 6. *For each cone C_i, let d_i be a vector parallel to the edge l_i such that $d_i + a_i$ is colinear to l_{i+1}.*

Proposition 2. *Consider the cone bounded by the lines l_i, l_{i+1} and associated to the vector a_i. We have*

- *The strips associated to the lines l_i and l_{i+1} are consecutive for the slopes and $V_i = a_i$.*

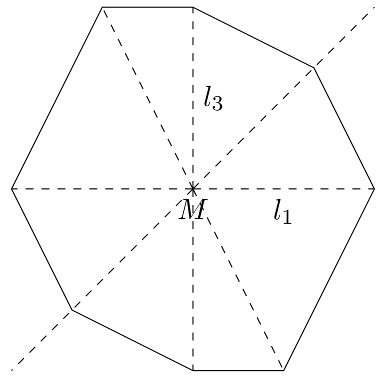

Figure 4: Polygon P^* associated to the quadrilateral $ABCD$

- *The vectors a_i, a_{i+1} have one vertex in common.*
- *The parallelogramm $\Sigma_i \cap \Sigma_{i+1}$ has a_i for diagonal and d_i for one side. The area of $\Sigma_i \cap \Sigma_{i+1}$ is equal to $|a_i \wedge d_i|$.*

Proof. Consider the cone C_i with boundaries l_i, l_{i+1} and a polygon $Q \in \mathcal{S}_P$. Let A be the vertex of Q associated to M. The first thing to remark is that the slope of the line (AM) is between the slopes of l_i and l_{i+1}. Thus A belongs to the edge parallel to l_{i+1} and the point $\tilde{T}A$ belongs to the edge parallel to l_i. This proves $V_i = a_i$ and the first point.

Consider one strip with vertices A, B, M, it means that M is the vertex that maximized the distance from (AB). By definition the polygon is in the strip between (AB) and $M + \mathbb{R}\vec{AB}$. Let N be the vertex neighbour of M in the polygon. Assume the vertex associated to (MN) is not B, denote it B'. The polygon is in the strip associated to M, N, B'. Thus this strip does not intersect the segment $[AB]$. Then the line (MN) has a slope bigger than (BB'). First part implies that in the ordering of the slopes, the slope of (MN) is the consecutive of the slope of (AB), contradiction.

The vector d_i is on the boundary of Σ_i by definition. Denote $a_i = v - w$ with v, w vertices of P. By the previous point, there exists a vertex w' such that ww' is on the boundary of Σ_{i+1}. Thus one side of $\Sigma_i \cap \Sigma_{i+1}$ is given by the line ww' and one side by the line d_i. The area of the parallelogramm is the cross product of one side by the diagonal. □

4.3 Comments

The preceding proposition may seem awkward, since we are not studying directly the outer billiard map to obtain results on its dynamics. Nevertheless we can transform the statement in terms of the outer billiard map T. The map T^2 is a piecewise translation, defined on several subsets of $\mathbb{R}^2$.

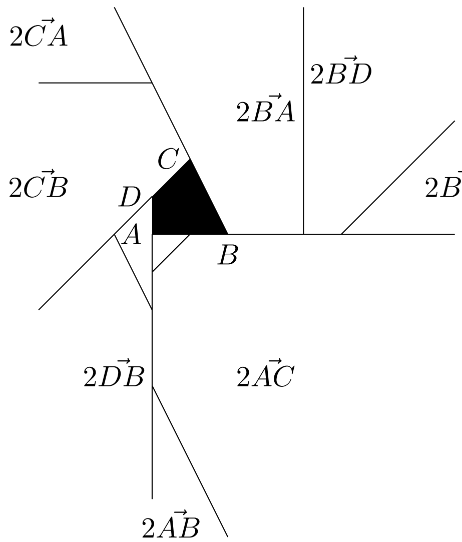

Figure 5: Definition of T^2

Some of them can be compact sets, see Figure 5. Outside a compact region containing the polygon, the sets are unbounded and the translation vectors are two by two opposite. The translation vectors are exactly the vectors V_i, see Proposition 2. The dynamics of T^2 is simple: A point m begins its trajectory by being translated by a vector V_i until it reaches another set where it moves by another vector V_i. Thus an orbit of point far away from P looks like the polygon P^*. The link between T^2 and the piecewise translations of vectors $V_1 \dots V_n$ can be extended to a neighborhood of P, but it is much more complicated, see the Pinwheel theorem [Sch11]. It is related in Proposition 1 to the case where A is closed to M. It is possible that the condition $A, \tilde{T}A \in C_i$ is not verified. This case is treated by the Pinwheel theorem in [Sch09].

5 Equivalence

5.1 Statement of results

The aim is to prove

Theorem 1. *The followings are equivalent:*

- $(\frac{v_1^*}{|a_1|}, \dots, \frac{v_n^*}{|a_n|})$ *are in* $\mathbb{PQ}^n$.
- $(|a_1 \wedge a_2|, \dots, |a_n \wedge a_1|)$ *are commensurates.*
- $(|\Sigma_2 \cap \Sigma_1|, \dots, |\Sigma_n \cap \Sigma_1|)$ *are commensurates.*

Remark 2. *The first point is the initial definition of a quasi-rational polygon given in [GS92]. The third is the definition by Schwartz in [Sch09].*

To do this we will prove the three following propositions. The theorem will be a clear consequence with help of Proposition 2.

Proposition 3. *The following are equivalent:*

- *There exists a rational solution* $(t_1, \ldots, t_n)$ *to*

$$\begin{cases} d_1 + a_1 = t_2 d_2 \\ d_1 + a_1 + t_2 a_2 = t_3 d_3 \\ d_1 + a_1 + \ldots t_n a_n = -d_1 \end{cases}$$

- $(|a_1 \wedge d_1|, \ldots, |a_n \wedge d_n|)$ *are commensurates.*

Proposition 4. *The following are equivalent:*

- *There exists a rational solution* $(t_1, \ldots, t_n)$ *to*

$$\begin{cases} d_1 + a_1 = t_2 d_2 \\ d_1 + a_1 + t_2 a_2 = t_3 d_3 \\ d_1 + a_1 + \ldots t_n a_n = -d_1 \end{cases}$$

- P *is a quasi-rational polygon.*

Proposition 5. *The following are equivalent:*

- *There exists a rational solution* $(t_1, \ldots, t_n)$ *to*

$$\begin{cases} d_1 + a_1 = t_2 d_2 \\ d_1 + a_1 + t_2 a_2 = t_3 d_3 \\ d_1 + a_1 + \ldots t_n a_n = -d_1 \end{cases}$$

- $(\frac{v_1^*}{|a_1|}, \ldots, \frac{v_n^*}{|a_n|})$ *are in* $\mathbb{PQ}^n$.

5.2 Proof of Proposition 3

The proof is based on Figure 6. Consider a polygon such that the points A, B, C, D, E are vertices labelled in such a way that the slopes of edges are in the increasing order AD, BC, BE. Also assume we have $a_1 = 2\vec{DC}$. Then Proposition 2 implies that $a_2 = 2\vec{BD}$. Let G be the intersection point of (AD) and (BC), and let H a point on the line (BC) such that (HD) is parallel to (BE). Then we have $d_1 = 2\vec{GD}, d_2 = 2\vec{HB}$. Moreover $\Sigma_1 \cap \Sigma_2$ is defined by the triangle GCD, and $\Sigma_3 \cap \Sigma_2$ is defined by BDH. Let r be the

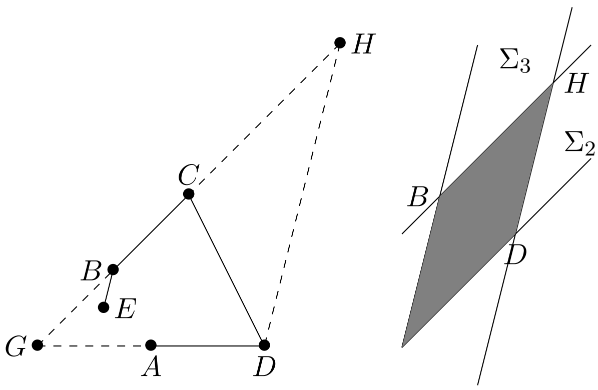

Figure 6: Proof of Proposition 3

real number such that $\vec{BH} = r\vec{GC}$. We see that Σ_1 is given by $((AD), 2\vec{DC})$, $\Sigma_2 = ((BC), 2\vec{BD})$. The intersection of the two strips Σ_1, Σ_2 has DC as a diagonal. A similar computation gives the intersection of the strips Σ_3, Σ_2. The two parallelograms are constructed on triangles BHD, GCD. We have

$$|BHD| = |\vec{BH} \wedge \vec{BD}|$$

$$|GCD| = |\vec{GC} \wedge \vec{CD}| = |\vec{GC} \wedge \vec{BD}| = r|BHD|.$$

The ratio of the areas of the two parallelograms is the same as the ratio of the area of triangles, thus it is equal to r. Thus we have proved:

$$r \in \mathbb{Q} \Longleftrightarrow \frac{|\Sigma_1 \cap \Sigma_2|}{|\Sigma_2 \cap \Sigma_3|} \in \mathbb{Q}.$$

Note that r is equal to the inverse of t_2 in the first part of Proposition. The proof of Proposition follows by induction.

5.3 Proof of Proposition 4

The proof is based on Figure 7. Consider an edge AD, and the associated vector $a_1 = 2\vec{DC}$. By Proposition 2 the vector a_2 is equal to $2\vec{BD}$ with BC edge of the polygon, and $a_3 = 2\vec{FB}$, with DF edge of the polygon. Let us call G the intersection of (CB) with (AD), and H the intersection of (CB) and (DF). Then $d_1 = 2\vec{GD}, d_2 = 2\vec{HB}$. Assume that the first item of Proposition 4 holds. Then there exist $r, r' \in \mathbb{Q}$ such that $\vec{GC} = r\vec{HB}, \vec{HD} = r'\vec{DF}$. Solving system shows that d_i is a rational linear sum of $a_1, \ldots, a_n$. By Proposition 2, the edges of P are rational combination of $a_1, \ldots, a_n$, the assumption implies: $\vec{GC} = q\vec{BC}, \vec{HD} = q'\vec{FD}$ with $q, q' \in \mathbb{Q}$. Now the relations $\vec{GC} = q\vec{BC} = r\vec{HB}$ imply $\vec{HB} = q"\vec{CB}$ with q'' rational number. For a point M, denote h_M the length of the orthogonal projection of M on (DB). The relations $\vec{HB} = q"\vec{CB}, \vec{HD} = q\vec{FD}$ gives $h_H = q"h_C$ and $h_F = qh_H$. By Proposition 2, the areas $|a_1 \wedge a_2|, |a_2 \wedge a_3|$ are given by areas of triangles BCD, DBF. The ratio of these areas is equal to the ratio between h_C and h_F. Thus the areas are commensurates. The other part of the proof is similar.

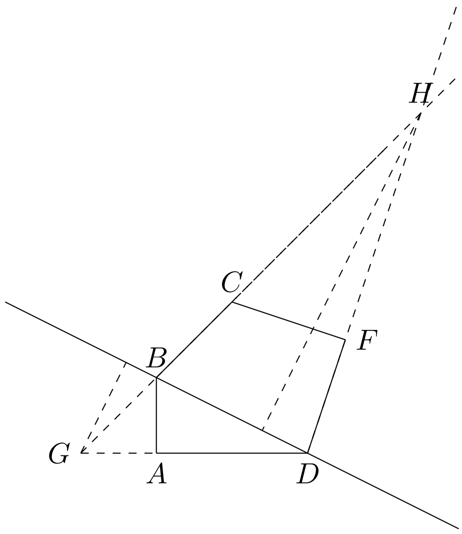

Figure 7: Proof of Proposition 4

5.4 Proof of Proposition 5

The proof is based on Figure 4. By Proposition 4 we know that the first statement is equivalent to the fact that P is a quasi-rational polygon. If the system has a rational solution, then the polygon defined by $d_1, d_1 + a_1, \ldots, d_1 + a_1 + \ldots t_n a_n$ is some polygon P^*. The edges $v_1^*, \ldots, v_n^*$ of this polygon are equal to $a_1, t_2 a_2, \ldots, t_n a_n$, thus the first implication is proved.

Conversly, consider the point M on l_1 such that $\vec{OM} = d_1$. By hypothesis, there exists a polygon with sides $r_1 a_1, \ldots r_n a_n$ with rational numbers $r_i, i \leq n$. Thus there exists an homothetic image of this polygon with vertex M, and all the edges fulfilling the same condition. This gives a rational solution of the system.

This proposition can be reformulated in

Corollary 1. *The polygon is quasi-rational if and only if : there exists* *rational numbers $t_2, \dots, t_n$ and $M \in \mathbb{R}^2$ such that*

$$\begin{cases} M \in l_1 \\ M + a_1 \in l_2 \\ M + a_1 + t_2 a_2 \in l_2 \\ M + a_1 + \dots + t_i a_i \in l_i \\ M + a_1 + \dots + t_n a_n = -M \end{cases}$$

6 All orbits are bounded for quasi-rational polygons

In this section we give a new proof of the following theorem using our definition of quasi-rational polygon. The aim is to understand the general outline of the proof, not to explain all the details.

Theorem 2. *[GS92] For a quasi-rational polygon, every orbit of the dual billiard map is bounded.*

We consider the unfolding and the cone C_1. We can tile periodically this cone by a parallelogramm with one side equal to d_1 and one diagonal equal to a_1. The same thing can be done in all cones. Consider a point x and the first hitting map with the next cone: f_1. We have $f_1(x+d_1) = f_1(x)+a_1+d_1$, we deduce $f_2(f_1(x+d_1)) = f_2(f_1(x)+t_2d_2)$. Since the polygon is quasi-rational there exists an integer n such that $f_2(f_1(x+nd_1)) = f_2(f_1(x)+nt_2d_2) = f_2(x)+n'(a_2+d_2)$. Now the first return map to the cone C_1 is the composition of $f_1, \dots, f_n$. We obtain that there exists a vector u such that for every x

$$F(x+u) = F(x)+u$$

In term of parallelograms, it means that we consider a point in one box and take the image of the box by F. We have a second periodic tiling of the cone by a parallelogram with side u and diagonal a_1. The orbit of the point x depends on the cutting of a box of the new tiling by the initial one. If the two tilings are commensurates then every orbit is bounded. We must compare u and d_1: they are rationally proportional by definition of quasi-rational polygon. If P is rational every box is mapped by F to a box, thus every orbit is periodic.

7 Graph of spokes

7.1 Definitions

By definition, for each integer i, a_i is a vector between two vertices of P, and every vertex is a starting point of some a_i. Define a graph with vertices the

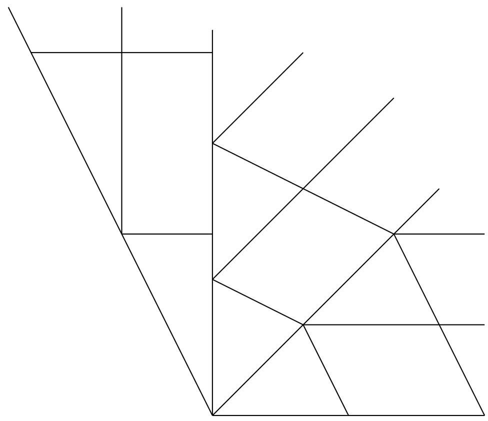

Figure 8: Tilings of consecutive cones

vertices of P, and there is an oriented edge starting from each vertex and joining the end of the vector a_i associated to the vertex. Denote it $\mathcal{S}(P)$, and we call it the **graph of spokes**.

Example 1. *Consider the polygon $ABCD$ of Figure 1, then $\mathcal{S}(P)$ is given by*

$$
\begin{tikzcd}
B \arrow[dr] & C \arrow[dl] \\
A \arrow[r] & D \arrow[u]
\end{tikzcd}
$$

Figure 9: Graph of spokes

Lemma 1. *This graph has following properties:*

- *Each vertex has an outgoing edge.*
- *Two edges can not have the same vertices.*
- *The graph contains a cycle.*

Proof. Proof left to the reader for the two first items. An edge between vertices A, B implies that some vector $a_i = \vec{AB}$. Thus the graph is the same thing as a map defined on the set of vertices. This map is defined everywhere but not necessarily injective. It is injective on a subset. On this subset the graph is a cycle. $\square$

Corollary 2. *For every polygon, there exists a rational relation between the vectors $a_1, \ldots, a_n$.*

Proof. By preceding Lemma there exists a cycle in the graph. It implies that the sum of vectors a_i associated to this cycle is null. □

Remark 3. *The notion of spokes is used in outer billiard by Schwartz to prove its result on the pinwheel map, see [Sch10].*

We now use the preceding tools to obtain new results on quasi-rational polygons.

8 Description of quasi-rational polygons

Theorem 3. *We have:*

- *A quadrilateral is a quasi-rational polygon if and only if it is rational.*
- *There exists a non regular and non rational quasi-rational pentagon.*
- *Assume the graph of spokes is a cycle (or an union of cycles). Then the polygon is quasi-rational.*

Proof.

- Consider (a_1, a_2) as a basis of $\mathbb{R}^2$, denote α, β the coordinates of a_3 in this basis, and (γ, δ) those of a_4: $a_3 = \alpha a_1 + \beta a_2, a_4 = \gamma a_1 + \delta a_2$. The numbers $|a_1 \wedge a_2|, |a_3 \wedge a_2|, |a_4 \wedge a_3|, |a_4 \wedge a_1|$ are proportional to $1, \alpha, \alpha\delta - \beta\gamma, \delta$. If the polygon is quasi-rational, by Theorem 1 we deduce $\alpha, \delta, \alpha\delta - \beta\gamma \in \mathbb{Q}$. This implies $\beta\gamma \in \mathbb{Q}$. Now Corollary 2 implies that there exists a rational linear relation between $a_1, \dots, a_4$. This relation concerns at least three vectors. All possibilities imply $\beta, \gamma \in \mathbb{Q}$. Thus the polygon has vertices on a lattice.
- Consider four points on a lattice of $\mathbb{R}^2$. Denote these points A, B, C, E. We will contruct a point D such that the pentagon $ABCDE$ will be as required. It suffices to consider one point D outside the lattice. We can always choose D such that the spokes of the pentagon $ABCDE$ are associated to vectors $\vec{AC}, \vec{BE}, \vec{CE}, \vec{DB}, \vec{EA}$. Then the rational relation is $a_1 + a_3 + a_5 = 0$. There is no other rational relation by definition of D. Now we can express the vectors $a_1, \dots, a_5$ in the basis (a_1, a_2). By construction a_3, a_5 have rational coordinates. Then we can always choose D such that the area $|a_2 \wedge a_3|$ is rational. The constructed pentagon is quasi-rational.
- Now assume that the graph of spokes is an union of cycles. By Corollary 1 a polygon is quasi-rational if for every side l_i, there exists $\lambda \in \mathbb{R}$ and rational numbers $r_1, \dots, r_n \in \mathbb{Q}^*$ such that

$$\lambda l_i + r_1 a_1 + \cdots + r_n a_n = 0.$$

If the graph is a union of cycles, then the map defined on vertices associated to the graph of spokes is invertible. It means that each vertex is a linear combination of $a_1 \dots a_n$. Thus the side l_i can be expressed as rational combination of the a_i's. Thus P is quasi-rational if there exists $r_1 \dots r_n \in \mathbb{Q}$ and $\lambda \in \mathbb{R}$ such that:

$$\lambda \sum r'_j a_j + r_1 a_1 + \cdots + r_n a_n = 0.$$

Since the graph is a cycle, there exists a rational relation between $a_1 \dots a_n$. Thus we can solve the equation and find $r_1 \dots r_n, r'_1 \dots r'_n$. □

Remark 4. *Consider the example of graph in Figure 9. In this case the preceding map is not a bijection since no edge goes to B.*

For regular polygon with odd number of sides (greater than five), the graph is not simply connected.

9 Remarks

9.1 Polygon with parallel sides

If the polygon has parallel sides, then the definition of [GS92] still works. Nevertheless the number of cones decreases. For the definition of [Sch09] we need to be more precise to define a strip. In this case two consecutive strips can have an intersection with infinite area. Thus the new definition of quasi-rational is that, up to a factor, the areas of $\Sigma_i \cap \Sigma_{i+1}$ are in $\mathbb{Z} \cup \{\infty\}$ for every integer i.

9.2 Regular polygons

A regular polygon with n edges is invariant by rotation of angle $2\pi/n$. Let ω be a n th root of unity, we have $a_i = \omega a_{i-1} + a_{i-2}$ for every integer i. Thus it is clear that $|a_i \wedge a_{i+1}|$ is a constant number, and a regular polygon is a quasi-rational polygon. Moreover the graph of spokes is a cycle, since the spoke a_{i+1} is the image of a_i by rotation of angle $2\pi/n$. This gives another proof of previous fact.

The study of regular polygons has been done if the number of sides is equal to 5 by Tabachnikov, see [Tab95b]. A description of the symbolic dynamics has been made for regular polygons with $3, 4, 5, 6, 10$ edges in [BC11]. In [Sch10] Schwartz initiates a study of the regular octogon.

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